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MATHEMATICAL MODELING OF NONLINEAR PROCESSES OF HEATING (COUNTERFLOW AND DIRECT-FLOW HEAT EXCHANGE) IN METALLURGY AND MACHINE CONSTRUCTION

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Solutions of the problems of counterflow and direct-flow heating of metal constructed on the solutions (obtained by the authors) of the corresponding nonlinear boundary-value heat-conduction problems are presented. An example illustrating the acceptable accuracy of the procedure proposed is given.

Introduction. The area of industrial heaters for heat and thermal treatment of ingots, billets, and parts is highly diverse. It includes soaking pits, chamber and continuous furnaces (pusher-type, mechanized-hearth, and ring ones), annealing and heat-treating furnaces, drying ovens, etc. All these devices differ in structure, purpose, and principle of operation (first of all, the type and model of heat exchange). The problem of resources and fuel saving, raising the capacity, and improving the quality of heating (drying) is crucial for the whole variety of heat-generating units. Mathematical modeling of the thermal processes in furnaces is one line of attack of this problem.

As applied to chamber-type furnaces, quite a large set of adequate procedures for calculating their thermal performance is already available in metallurgical heat engineering. A different situation arises with the theory and practice of continuous furnaces employing the modes of direct-flow and counterflow heat exchange. In heat exchange of a material and a gas moving in opposition (concurrently), the temperatures of both agents change in close interdependence; one can establish it only by solving the corresponding boundary-value problem. The temperature of the heat-transfer agent (gas) is unknown in this problem, which requires that an additional boundary condition in the form of a heat-balance equation be introduced into the mathematical heat-conduction problem. This complication of the mathematical model substantially slows down the progress of the general technical theory of counterflow and direct-flow heat exchange.

Counterflow Heating of Thermally Massive Bodies with Variable Thermophysical Characteristics (by Radiation and Convection Simultaneously). The first effort to analytically describe the counterflow (direct-flow) heat exchange of thermally massive ingots under thermal radiation was made by A. V. Kavaderov as early as the midtwentieth century [1, 2]. The solutions obtained were based on two auxiliary functions determined by special tables that were composed using an analog computer — a hydrostatic integrator of D. V. Budrin's system.

A substantial breakthrough in investigations of combined counterflow radiative-convective heat exchange has been carried out by V. I. Timoshpol'skii, Yu. S. Postol'nik, Yu. A. Samoilovich, and others [3, 4], developed more comprehensively in [5, 6]. The only drawback of the solutions given in these works is that the thermophysical properties of a metal are taken to be constant, which can cause errors in calculations.

Developing the solutions obtained, the authors of the present work have found analytical solutions of the nonlinear boundary-value problem of heat conduction of thermosensitive massive bodies under combined (radiative-convective) counterflow heat exchange.

The mathematical model has the form

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$$\frac{1}{\rho^m}\frac{\partial}{\partial\rho}\left[\rho^m(1+\varepsilon_\lambda\theta)\frac{\partial\theta}{\partial\rho}\right]=\frac{\partial\theta}{\partial\tau}\,; \tag{1}$$

$$[1+\varepsilon_\lambda\theta_s(\tau)]\left.\frac{\partial\theta}{\partial\rho}\right|_{\rho=1}=\mathrm{Sk}\left\{[\theta_g^4(\tau)-\theta_s^4(\tau)]+\zeta\,[\theta_g(\tau)-\theta_s(\tau)]\right\}; \tag{2}$$

$$\left.\frac{\partial\theta}{\partial\rho}\right|_{\rho=0}=0\,; \tag{3}$$

$$\frac{\partial\theta_g}{\partial\tau}=\mathrm{Sk}\left\{[\theta_g^4(\tau)-\theta_s^4(\tau)]+\zeta\,[\theta_g(\tau)-\theta_s(\tau)]\right\}n_m\,; \tag{4}$$

$$\theta(\rho,0)=\theta_0=\theta'=\mathrm{const}\,,\quad \theta_g(0)-\theta''_g=1\,, \tag{5}$$

where we have introduced the dimensionless quantities

$$\theta(\rho,\tau)=T(\rho,\tau)/T''_g,\quad \rho=r/R\,,\quad \tau=at/R^2\,,\quad \varepsilon_\lambda=\delta_\lambda T''_g/\lambda_0\,,$$
$$\mathrm{Sk}=\sigma_v/T''^3_g R/\lambda_0\,,\quad \mathrm{Bi}=\alpha R/\lambda_0\,,\quad \zeta=\frac{\mathrm{Bi}}{\mathrm{Sk}}\,,\quad n=\frac{Vc}{V_g c_g}\,,\quad n_m=(1+m)\,n\,. \tag{6}$$

The earlier investigations [7] have shown that the variability of the thermal conductivity $\lambda(T)$ exerts the most substantial influence on the conductive process. In this case in thermal calculations, one usually takes (see, e.g., [8, 9]) the function $\lambda(T)$ to be linear and the remaining thermophysical characteristics to be constant, i.e.,

$$\lambda(T)=\lambda_0+\varepsilon_\lambda T=\lambda_0\left(1+\frac{\delta_\lambda T''_g}{\lambda_0}\theta\right)=\lambda_0(1+\varepsilon_\lambda\theta)\,;\quad c=c(\bar{T})=\mathrm{const}\,;\quad a=a(\bar{T})=\mathrm{const}\,,$$

which has been taken in the mathematical model (1)–(6).

The widely used [10] thermal-layer model has been adopted as the organizer method of solution of the nonlinear boundary-value heat-conduction problem (1)–(6) formulated, whereas the well-known method of equivalent sources of Yu. S. Postol'nik [7–11] has been used for direct realization of the process of solution; this method showed itself well in solving highly diverse linear and nonlinear heat-conduction problems and was further developed theoretically and applied to the investigation of high-temperature processes [4, 14, 15]. Also, it was tested earlier in linear counterflow problems [16, 17].

In the first (inertial) step ($0\le\tau\le\tau_0$, $\beta(\tau)\le\rho\le1$), the solution obtained has the form

$$\theta_1(\rho,\tau)=\theta'+\Delta\theta_1(\tau)\,[\rho-\beta(\tau)]^2/l^2(\tau)\,, \tag{7}$$

where $\Delta\theta(\tau)=\theta_{1s}(\tau)-\theta'$ is the temperature difference over the thickness $l(\tau)=1-\beta(\tau)$ of the warmed-up (thermal) layer;

$$l(\tau)=\sqrt{6(1+m)(1+\varepsilon_\lambda\theta')\,\tau}\,,\quad \tau_0=[6(1+m)(1+\varepsilon_\lambda\theta')]^{-1}\,; \tag{8}$$

$$\Delta\theta_1(\tau)=\tau/l(\tau)=\sqrt{\tau/[6(1+m)(1+\varepsilon_\lambda\theta')]}\,,\quad \Delta\theta_1^0=\tau_0\,; \tag{9}$$

$$\theta_{1g}(\tau) = 1 + \frac{n\tau}{3}\left[1 + \frac{\varepsilon_\lambda l(\tau)}{9(1+m)(1+\varepsilon_\lambda\theta')^2}\right], \quad \theta_{1g}(\tau_0) = \theta_{1g}^0 . \tag{10}$$

In the second (ordered) step ($\tau_0 \le \tau \le \tau_*$, $0 \le \rho \le 1$), we have

$$\theta_2(\rho, \tau) = \frac{1}{\varepsilon_\lambda}\left\{\sqrt{[1+\varepsilon_\lambda\theta_{2s}(\tau)]^2 - \varepsilon_\lambda \mathrm{Sk}\left\{[\theta_{2g}^4(\tau) - \theta_{2s}^4(\tau)] + \zeta[\theta_{2g}(\tau) - \theta_{2s}(\tau)]\right\}(1-\rho^2)} - 1\right\} . \tag{11}$$

The gas temperature $\theta_{2g}(\tau)$ is determined by the transcendental equation

$$\Phi_g(\tau) - \Phi_g^0 = 4(1+m)\,\theta_{\mathrm{env}} k_1 \,\mathrm{Sk}\,(\tau - \tau_0)/k_2^2 , \tag{12}$$

where

$$\Phi_g(\tau) = \ln \theta_{2g}(\tau) - p \ln[1 - k_2\theta_{2g}(\tau)] + (p-1)\ln[1 + \varepsilon_\lambda\theta_{2g}(\tau)] - \frac{1 + k_2\theta_{2g}(\tau)}{2k_2^2\theta_{2g}^2(\tau)} ; \tag{13}$$

$$p = 1 + 4k_1\,\mathrm{Sk}/[(3+m)(k_2 + \varepsilon_\lambda)\,k_2^2] , \quad k_1 = 1 + \zeta\frac{0.275 + 0.058m}{\mathrm{Sk}} , \quad k_2 = (1-n)/\theta_{\mathrm{env}} ,$$
$$\theta_{\mathrm{env}} = 1 - n\theta' - \frac{2n}{3+m}[m\tau_0 + 2\varepsilon_\lambda(3+m)] . \tag{14}$$

Knowing the gas temperature and the metal-surface temperature $\theta_{2s}(\tau)$, we find the solution of the algebraic equation

$$\theta_{2s}^4(\tau) + a_{2s}\theta_{2s}^2(\tau) + a_{1s}(\tau)\,\theta_{2s}(\tau) = a_{0s}(\tau) , \tag{15}$$

$$a_{2s} = \varepsilon_\lambda(h + \zeta) , \quad a_{1s}(\tau) = (h+\zeta)\left\{1 + \varepsilon_\lambda[\theta_{\mathrm{env}} - \theta_{2g}(\tau)/n]\right\} ;$$
$$a_{0s}(\tau) = \theta_{2g}^4(\tau) + h[(1 + n\zeta/h)\,\theta_{2g}(\tau) - \theta_{\mathrm{env}}] , \quad h = \frac{3+m}{\mathrm{Sk}} . \tag{16}$$

Obtaining the values of the temperatures $\theta_{2s}(\tau)$ and $\theta_{2g}(\tau)$, by solution of (11) we compute the temperature of the center $\theta_{2\mathrm{center}}(\tau)$, the temperature difference $\Delta\theta_2(\tau) = \theta_{2s}(\tau) - \theta_{2\mathrm{center}}(\tau)$, and the average (over the volume) temperature $\theta_2(\tau) = \theta_{2s}(\tau) - \frac{2}{3+m}\Delta\theta_2(\tau)$.

The time of completion of heating τ_* is determined by solution of (13) under the assumption that $\theta_{2s}(\tau_*) = \theta_{2s}^* = \eta\theta_{2g}^*$. Substituting $\theta_{2s}^* = \eta\theta_{2g}^*$ into (15) and (16), we arrive at an algebraic equation similar to (15) but for θ_{2g}^* now. The new coefficients have the form

$$a_{2g} = \varepsilon_\lambda\eta h_* , \quad a_{1g} = h_*\left(1 + \frac{\zeta}{h_*}\frac{1-\eta}{1-\eta^4} - \frac{\varepsilon_\lambda\eta\theta_{\mathrm{env}}}{1-n\eta}\right), \quad a_{0g} = \frac{h_*\theta_{\mathrm{env}}}{1-\eta n} , \quad h_* = h(1-n\eta)/[n(1-\eta^4)] . \tag{17}$$

Knowing θ_{2g}^*, we determine, from expressions (13) and (14), the heating time τ_*:

$$\tau_* = \tau_0 + k_2^2(\Phi_g^* - \Phi_g^0)/[4(1+m)\,k_1\,\mathrm{Sk}\,\theta_{\mathrm{env}}] . \tag{18}$$

Thus, the formulated nonlinear boundary-value problem of counterflow heating of thermally massive bodies of basic (classical) geometry is completely solved.

To evaluate the adequacy of the constructed analytical model to the results of analog and numerical modeling we have checked the numerical example borrowed from [2]. The calculation results confirmed the data of the previous investigations [6, 18] and showed that calculations by the method of equivalent sources and the data of the analog computer [2] were in complete agreement, in practice. The same high convergence of the results has also been confirmed [19] in comparison with numerical (finite-difference) methods.

Solution of the Problem of Heating of Prismatic Ingots and Billets in the Counterflow Regime (Two-Dimensional Model). Continuous walking-beam and walking-hearth furnaces in which heated ingots and billets are arranged with a gap have enjoyed wide application in the metallurgical industry recently. When the processes of heating of metal in such furnaces are modeled, it becomes necessary to solve the counterflow boundary-value problem, which enables one to improve the accuracy of the results obtained [20, 21].

As a continuation of the earlier investigations [4, 22], we have obtained below the solution of the problem of heat conduction and thermoelasticity in heating of prismatic billets in the counterflow regime. The mathematical model of the heating has the form

$$a\left[\frac{\partial^2 T}{\partial x_1^2}+\frac{\partial^2 T}{\partial x_2^2}\right]=\frac{\partial T}{\partial t}\,; \tag{19}$$

$$\lambda \left.\frac{\partial T}{\partial x_i}\right|_{x_i=H_i}=\alpha\,[T_{\mathrm{g}}(t)-T(x_1,x_2,t)]\Big|_{x_i=H_i}\,,\quad \left.\frac{\partial T}{\partial x_i}\right|_{x_i=0}=0\,; \tag{20}$$

$$V_{\mathrm{g}}(\tau)\,c_{\mathrm{g}}\frac{\partial T_{\mathrm{g}}}{\partial t}=\alpha(t)\,[T_{\mathrm{g}}(t)-\overline{T}_{\mathrm{s}}(t)]\,2A-c_{\mathrm{g}}\,(T_{\mathrm{comb}}-T_{\mathrm{g}}(t))\left|\frac{dV_{\mathrm{g}}(\tau)}{dt}\right|\,; \tag{21}$$

$$T(x_1,x_2,0)=T'=T_0=\mathrm{const}\,,\quad T_{\mathrm{g}}(0)=T''_{\mathrm{g}}=\mathrm{const}\,. \tag{22}$$

To allow for radiative-convective heat exchange and to minimize difficulties associated with obtaining the solution of the heat-conduction problem with nonlinear boundary conditions of the third kind we have introduced the total heat-transfer coefficient $\alpha=\alpha_{\mathrm{c}}+\alpha_{\mathrm{r}}$, where $\alpha_{\mathrm{r}}=\sigma_{\mathrm{v}}(T_{\mathrm{g}}+T_{\mathrm{s}})(T_{\mathrm{g}}^2+T_{\mathrm{s}}^2)$. Passing to dimensionless quantities

$$\theta(\xi_1,\xi_2,\tau)=\frac{T(x_1,x_2,t)-T_0}{T''_{\mathrm{g}}-T_0}\,,\quad \xi_1=\frac{x_1}{H_1}\,,\quad \xi_2=\frac{x_2}{H_2}\,,\quad \tau=\frac{at}{H_1^2}\,,$$

$$\mathrm{Bi}=\frac{\alpha H}{\lambda}\,,\quad N(\tau)=2n(\tau)=2V_{\mathrm{m}}c_{\mathrm{m}}/V_{\mathrm{g}}(\tau)\,c_{\mathrm{g}}=\frac{2AHc_{\mathrm{m}}}{V_{\mathrm{g}}(\tau)\,c_{\mathrm{g}}}=\frac{2H2HLc_{\mathrm{m}}}{V_{\mathrm{g}}(\tau)\,c_{\mathrm{g}}}\,, \tag{23}$$

we obtain

$$\frac{\partial^2\theta}{\partial\xi_1^2}+\frac{\partial^2\theta}{\partial\xi_2^2}=\frac{\partial\theta}{\partial\tau}\,; \tag{24}$$

$$\left.\frac{\partial\theta}{\partial\xi_i}\right|_{\xi_i=1}=\mathrm{Bi}(\tau)\,[\theta_{\mathrm{g}}(\tau)-\theta(\xi_1,\xi_2,\tau)]\Big|_{\xi_i=1}\,,\quad \left.\frac{\partial\theta}{\partial\xi_i}\right|_{\xi_i}=0\,; \tag{25}$$

$$\frac{\partial \theta_g(\tau)}{\partial \tau} = \text{Bi}(\tau)\,[\theta_g(\tau) - \bar{\theta}_s(\tau)]\,N(\tau) - (\theta_{\text{comb}} - \theta_g(\tau))\,\frac{dV_g(\tau)}{d\tau} / V_g(\tau)\,; \tag{26}$$

$$\theta(\xi_1, \xi_2, 0) = \theta' = \theta_0 = 0\,, \quad \theta_g(0) = \theta''_g = 1\,. \tag{27}$$

Under the assumption that the gas flow rate changes discretely, the second term in Eq. (26) vanishes and the change in the flow rate of flue gases is allowed for by the ratio of water equivalents [23, 24]. Allowing for the transient character of the first (inertial) step, we consider the second (ordered) step of heating. In accordance with this, we replace the initial conditions in the mathematical model (23)–(27) by the following conditions:

$$\theta_2(\xi_1, \xi_2, \tau)\Big|_{\substack{\xi_1=0\\ \xi_2=0}} = \theta_{\text{center}}(\tau_0) = 0\,, \quad \theta_g(\tau_0) = \theta''_g = 1\,. \tag{28}$$

The time of warming-up of the prism τ_0 can be determined from the formula

$$\frac{1}{\tau_0} = \frac{1}{\tau_{01}} + \frac{1}{\tau_{02}} \quad \text{or} \quad \tau_0 = \frac{\tau_{01}\tau_{02}}{\tau_{01} + \tau_{02}}\,, \tag{29}$$

where $\tau_{0i} = 1/6$ is the time of symmetric warming-up of an unbounded plate of thickness $2H$ along the ith coordinate. For a square prism, formula (29) yields $\tau_0 = 0.083$. The solution of problem (23)–(27) in a regular heating step has the form

$$\theta_2(\xi_1, \xi_2, \tau) = \theta_g(\tau) - \frac{f_2(\tau)}{2}\left\{\left[\frac{2 + \text{Bi}(\tau)}{\text{Bi}(\tau)} - \frac{\xi_1^2 + \xi_2^2}{2}\right]\right.$$

$$\left.- \frac{\text{Bi}(\tau)}{2\Phi(\tau)}\left[\frac{(2 + \text{Bi}(\tau))\,[\cosh(\mu(\tau)\,\xi_1) + \cosh(\mu(\tau)\,\xi_2)]}{\text{Bi}(\tau)} - \xi_1^2\cosh(\mu(\tau)\,\xi_2) + \xi_2^2\cosh(\mu(\tau)\,\xi_1)\right]\right\}; \tag{30}$$

$$\theta_g(\tau) = 1 + \int_{\tau_0}^{\tau} f_2(\eta)\,\Psi(\eta)\,d\eta\,, \tag{31}$$

where we have introduced the notation

$$\Psi(\tau) = \frac{\text{Bi}(\tau)\,N(\tau)}{2}\left[\frac{(6 + \text{Bi}(\tau))}{3\text{Bi}(\tau)} - \frac{\mu(\tau)\cosh(\mu(\tau))\,(5 + 3\text{Bi}(\tau)) + 6\sinh(\mu(\tau))}{6\mu(\tau)\,[\text{Bi}(\tau)\cosh(\mu(\tau)) + \mu(\tau)\sinh(\mu(\tau))]}\right];$$

$$\Phi(\tau) = \text{Bi}(\tau)\cosh(\mu(\tau)) + \mu(\tau)\sinh(\mu(\tau))\,; \quad \mu(\tau) = \sqrt{\frac{3\text{Bi}(\tau)}{3 + \text{Bi}(\tau)}}\,; \quad M(\tau) = \frac{1}{\mu^2(\tau)} - \frac{(6 + 2\text{Bi}(\tau))\sinh(\mu(\tau))}{6\mu(\tau)\,\Phi(\tau)}\,;$$

$$f_2(\tau) = f_2^0\exp\left[-\int_{\tau_0}^{\tau}\frac{\frac{d}{d\eta}M(\eta) + 1 - \Psi(\eta)}{M(\eta)}\,d\eta\right];$$

$$f_2(\tau_0) = f_2^0 = \frac{2\text{Bi}(\tau_0)\,[\text{Bi}(\tau_0)\cosh(\mu(\tau_0)) + \mu(\tau_0)\sinh(\mu(\tau_0))]}{(2 + \text{Bi}(\tau_0))\,[\text{Bi}(\tau_0)\cosh(\mu(\tau_0)) + \mu(\tau_0)\sinh(\mu(\tau_0)) - \text{Bi}(\tau_0)]}\,.$$

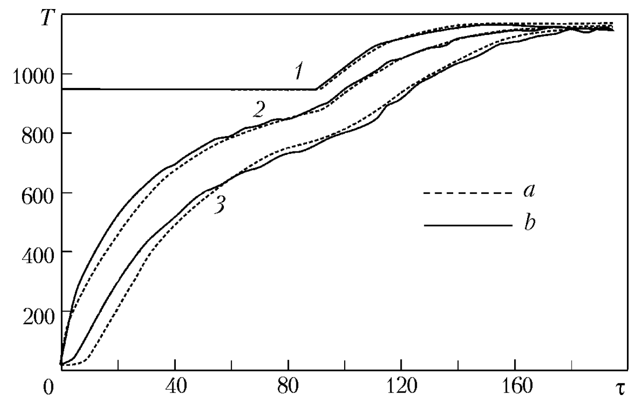

Fig. 1. Comparison of the experimental values of the temperatures of the gas (1), the surface (2), and center of a billet (3) to the calculated data on the developed mathematical model of counterflow heat exchange: a) modeling results; b) experiment. T, $^{\circ}$C; τ, min.

With the aim of identifying the solution obtained we compared the calculated data to the results of full-scale experiment. The balance relation (21) is replaced by the equation

$$\text{at}\quad t \le t_1 \quad T_g = \text{const} = T_g'' = 950^{\circ}\text{C}\ ;$$

$$\text{at}\quad t > t_1 \quad V_g(\tau)\, c_g \frac{\partial T_g}{\partial t} = \alpha(t)\,[T_g(t) - \overline{T}_s(t)]\, 2A - c_g\,(T_{\text{comb}} - T_g(t)) \left| \frac{dV_g(\tau)}{dt} \right|$$

and accordingly Eq. (26), in dimensionless form, appears as

$$\text{at}\quad \tau \le \tau_1 \quad \theta_g = \theta(\tau_0) = \theta_g'' = 1\ ;$$

$$\text{at}\quad \tau > \tau_1 \quad \frac{\partial \theta_g}{\partial \tau} = \text{Bi}(\tau)\,[\theta_g(\tau) - \overline{\theta}_s(\tau)]\, N(\tau) - (\theta_{\text{comb}} - \theta_g) \frac{dV_g(\tau)}{d\tau} / V_g(\tau)\ ,$$

where t_1 and τ_1 are respectively the dimensional and dimensionless time of holding of a billet in the beginning of the continuous zone.

Another feature of the model developed is that the cross section of the heated billet is taken to be square, which has been reflected in the formula for the temperature field (30). Therefore, in comparing with experimental data (billet cross section 250 × 300 mm), we have used an "equivalent square" cross section equal to 274 × 274 mm in area.

In calculation of the heating of a square (prismatic) billet with an equivalent cross section of 0.274 × 0.274 mm and of length 5.3 m in the 850-mill furnace, we took the following initial data: heat capacity of the metal c_m = 650 J/(kg·K), thermal conductivity λ = 35 W/(m·K), thermal diffusivity $a = 6.9 \cdot 10^{-6}$ m^2/sec, and heat capacity of the combustion products $\overline{c}$ = 1300 J/(m^3·K). The flow rate of natural gas and air was taken from the readings of regular meters during the experiment. The water equivalent, from formula (23), is N = 1.6.

Figure 1 compares the temperature field of the billet and the temperature of the heat-transfer agent (30) and (31) to experimental data. The value of the total heat-transfer coefficient was taken according to [25] where the convective component was within 40–60 W/(m^2·K). It is clear from the figure that the disagreement is no higher than 4% even in the zone of phase transitions (metal temperature of the order of 700–750°C).

Thus, the solution obtained can be used for determination of the thermally stressed state of prismatic billets in heating with the aim of developing resource-saving regimes.

Direct-Flow Convective Heating. Industrial units operating on the principle of direct-flow heat exchange are of particular interest in engineering calculations in certain cases. Among these units are, first of all, heat exchangers of the metallurgical and machine-construction industries that operate with high-temperature waste gases and employ (totally or partially) this mode of heat exchange because of the hazard of failure of structures due to the intense oxidation and decarbonization, and furnaces of high-speed heating of metal.

In mathematical modeling of direct-flow heat exchange, we consider the corresponding problem in the following formulation [26]:

$$\frac{1}{\rho^m} = \frac{\partial}{\partial \rho}\left[\rho^m \frac{\partial \theta}{\partial \rho}\right] = \frac{\partial \theta}{\partial \tau} ; \tag{32}$$

$$\left.\frac{\partial \theta}{\partial \rho}\right|_{\rho=1} = \mathrm{Bi}\,[\theta_g(\tau) - \theta_s(\tau)] , \quad \left.\frac{\partial \theta}{\partial \rho}\right|_{\rho=0} = 0 ; \tag{33}$$

$$\frac{d\theta_g}{d\tau} = -\,\mathrm{Bi}\,[\theta_g(\tau) - \theta_s(\tau)]\, n_m ; \tag{34}$$

$$\theta(\rho, 0) = \theta_0 = 0 , \quad \theta_g(0) = \theta_g^0 = 1 . \tag{35}$$

In the inertial step ($0 \le \tau \le \tau_0$), just as in [27, 28], we use the worked-out solution of the method of equivalent sources [7]

$$\theta_1(\rho, \tau) = \frac{\theta_{g1}(\tau)\,\mathrm{Bi}}{[2 + \mathrm{Bi}\, l(t)]\, l(\tau)} [\rho - \beta(t)]^2 , \quad \beta(\tau) \le \rho \le 1 , \tag{36}$$

where the thickness $l(\tau) = 1 - \beta(\tau)$ of the thermal layer in [27] is represented by the formula

$$l(\tau) = \sqrt{6(1 + m)\, k\tau} , \quad k = (3 + 2\mathrm{Bi})/(3 + \mathrm{Bi}) . \tag{37}$$

The temperature gas function is

$$\theta_{g1}(\tau) = \exp\left\{-M\left[l(\tau) - \frac{2}{\mathrm{Bi}} \ln\left(1 + \frac{\mathrm{Bi}\, l(\tau)}{2}\right)\right]\right\} , \tag{38}$$

where

$$M = \frac{2n}{3k} . \tag{39}$$

For $\tau = \tau_0 = (3 + \mathrm{Bi})/[6(1 + m)(3 + 2\mathrm{Bi})]$, when the warming-up step is completed, we have $l(\tau_0) = 1$; the temperatures of the body and the gas are determined by the expressions

$$\theta_1(\rho, \tau_0) = \theta_1^0(\rho) = \frac{\theta_{g1}^0 \mathrm{Bi}}{2 + \mathrm{Bi}} \rho^2 , \tag{40}$$

$$\theta_{g1}(\tau_0) = \theta_{g1}^0 = \exp\left\{-M\left[1 - \frac{2}{\mathrm{Bi}} \ln\left(1 + \frac{\mathrm{Bi}}{2}\right)\right]\right\} . \tag{41}$$

In the ordered step of heating ($\tau \ge \tau_0$), we have obtained the following solutions:

the temperature of the heat-transfer agent is

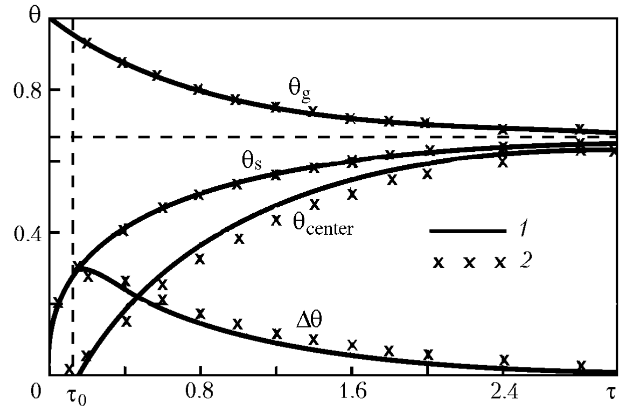

Fig. 2. Change in the dimensionless temperatures of the surface θ_s and center of the plate θ_{center}, the gas temperature θ_g, and the temperature difference over the cross section of the plate $\Delta\theta$ heated in counterflow as a function of the dimensionless time for the data of (48): 1) approximate solution by the method of equivalent sources; 2) exact solution [1].

$$\theta_{g2}(\tau) = \theta_{g1}^0 \left\{1 - D\,[1 - \Phi(\tau)]\right\}, \tag{42}$$

where

$$\Phi(\tau) = \exp[-\mu(\tau - \tau_0)]\,; \quad \mu = \frac{(1+m)\,\mathrm{Bi}}{1 + \mathrm{Bi}/(3+m)}(1+n)\,; \quad D = \frac{2n\,[1 + \mathrm{Bi}/(3+m)]}{(2+\mathrm{Bi})(1+n)}\,; \tag{43}$$

the temperature field of the metal is

$$\theta_2(\rho, \tau) = \theta_{g1}^0 \left\{1 - D - \left[1 - D - \frac{\mathrm{Bi}}{2+\mathrm{Bi}}\rho^2\right]\Phi(\tau)\right\}. \tag{44}$$

Setting $\rho = 1$ and $\rho = 0$ in (44), we obtain the temperature functions of the surface and center of the body

$$\theta_{s2}(\tau) = \theta_{g1}^0 \left[1 - D - \left(\frac{2}{2+\mathrm{Bi}} - D\right)\Phi(\tau)\right], \tag{45}$$

$$\theta_{center2}(\tau) = \theta_{g1}^0 (1 - D)\,[1 - \Phi(\tau)]\,. \tag{46}$$

In heat-engineering calculations, knowledge of the mass-mean temperature of the body

$$\bar{\theta}(\tau) = (1+m)\int_0^1 \theta_2(\rho, \tau)\,\rho^m d\rho\,, \quad \bar{\theta}(\tau) = \theta_{g1}^0 \left\{1 - D - \left[1 - D - \frac{1+m}{3+m}\frac{\mathrm{Bi}}{2+\mathrm{Bi}}\right]\Phi(\tau)\right\} \tag{47}$$

is sometimes necessary. To evaluate the exactness of the solutions obtained we carried out a numerical experiment with the example borrowed from [26]:

$$m = 0\,, \quad \mathrm{Bi} = 1\,, \quad n = 0.5\,. \tag{48}$$

The results of computations from the approximate (40)–(42), (44) and exact [26] solutions are presented in Fig. 2 whence it is clear that the exactness of the solution proposed here is quite acceptable for engineering calculations. Any disagreement between them is easily compensated by the exceptional simplicity of the latter.

Conclusions. The approximate solutions obtained for the problems of heating of moving billets or ingots in a gas counterflow and direct flow possess simplicity and exactness sufficient for practice (which has been confirmed by comparisons to experimental data) and can efficiently be used for calculations of thermophysical and heat-exchange processes in metallurgy and machine construction (continuous pusher-type furnaces, walking-hearth and walking-beam furnaces, continuous furnaces of high-speed heating of metal, and heat exchangers).

NOTATION

A, area of the lateral prism surface, m^2; a, thermal diffusivity, m^2/sec; a_{0s}, a_{1s}, and a_{2s}, coefficients of the algebraic equation on determination of the metal-surface temperature; Bi, Biot number; c and c_g, heat capacities of the metal and the gas respectively, J/(kg·K) or J/(m^3·K); D, integration constant; f_2 and f_2^0, auxiliary functions for determination of the metal temperature; H_1 and H_2, half-thicknesses of the prism, m; h, integration constant; k, k_1, and k_2, integration constants; L, billet length, m; l, dimensionless thickness of the warmed-up layer; m, auxiliary function for determination of the metal temperature; m, combining shape parameter of a body ($m = 0$, plate, $m = 1$, cylinder, and $m = 2$, sphere); N, ratio of the water equivalents of the metal and the gas in a two-dimensional problem; n, ratio of the water equivalents of the metal and the gas; n_m, refined value of the ratio of the water equivalents, combined with allowance for the shape factor; p, integration constant; R, half-thickness of a plate or the radius of a cylinder (sphere), m; r, running coordinate, m; Sk, Stark number, m; T, temperature, K; T_m', T_m'', T_g', and T_g'', metal and gas temperatures at the inlet and outlet of the furnace respectively, K; t, time, sec; V and V_g, flow rates of the metal and the gas, m and m^3/sec; x_1, x_2, coordinates of a body, m; α, coefficient of heat transfer by convection, W/(m^2·K); β, dimensionless thickness of the not warmed-up layer; δ_λ, change in the thermal conductivity, W/(m·K); ε_λ, relative change in the thermal conductivity; ζ, ratio of the Biot and Stark numbers; η, prespecified exponent of completeness of the heating; $\theta(\rho, \tau)$, dimensionless metal temperature; λ, thermal conductivity, W/(m·K); μ, integration constant; ξ_1 and ξ_2, dimensionless coordinates of the prism; $\Delta\theta$, temperature difference over the cross section; ρ, dimensionless coordinate reckoned from the center; σ_v, visible emissivity, W/(m^2·K^4); τ, dimensionless time (Fourier number); Φ_g, transcendental function for determination of the gas temperature; Φ_g^0, the same but at the beginning of the regular step; Ψ, parameter of integration. Subscripts and superscripts: v, visible; g, gas; comb, combustion; c, convective; r, radiant; m, material (solid body); s, surface of a body; env, environment; center, center of a body; 1, inertial step of heating; 2, ordered step; i, No. of coordinate of a body ($i = 1$, coordinate along the abscissa axis; $i = 2$, along the ordinate axis); 0, initial value; $'$ and $''$, inlet and outlet values respectively; *, completion of the process; $\bar{}$, average value.

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